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Risk minimization through portfolio replication

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Abstract. We use a replica approach to deal with portfolio optimization problems. A given risk measure is minimized using empirical estimates of asset values correlations. We study the phase transition which happens when the time series is too short with respect to the size of the portfolio. We also study the noise sensitivity of portfolio allocation when this transition is approached. We consider explicitely the cases where the absolute deviation and the conditional value-at-risk are chosen as a risk measure. We show how the replica method can study a wide range of risk measures, and deal with various types of time series correlations, including realistic ones with volatility clustering.

PACS. 89.65.Gh Economics; econophysics, financial markets, business and management

1 Introduction

The portfolio optimization problem dates back to the pioneering work of Markowitz [1] and is one of the main issues of risk management. Given that the input data of any risk measure ultimately come from empirical observations of the market, the problem is directly related to the presence of noise in financial time series. In a more abstract (model-based) approach, one uses Monte Carlo simulations to get "in-sample" evaluations of the objective risk function. In both cases the issue is how to take advantage of the time series of the returns on the assets in order to properly estimate the risk associated with our portfolio. This eventually results in the choice of the risk measure, and a long debate in the recent years has drawn the attention on two important and distinct clues: the mathematical property of *coherence* [2], and the noise sensitivity of the optimal portfolio. The rational behind the first of these issues lies in the need of a formal (axiomatic) translation of the basic common principles of risk management, like the fact that portfolio diversification should always lead to risk reduction. Moreover, requiring a risk measure to be coherent implies the existence of a unique optimal portfolio and a well-defined variational principle, of obvious relevance in practical cases. The second issue is also a very delicate one. In a realistic experimental set-up, the number N of assets included in a portfolio can be of order 10^2 to 10^3, while the length of a trustable time series hardly goes beyond a few years, i.e. $T \sim 10^3$. A good estimate of any extensive observable would require the condition $N/T \ll 1$ to hold, but this is rarely the case. Instead, the ratio of assets to data points, N/T, will be considered as a finite number.

Assuming a multinormal distribution of returns, numerical studies have shown the existence of a phase transition in the large N limit, at fixed N/T [4]. The ratio N/T plays the role of a control parameter. When it increases, there exists a sharply defined threshold value where the estimation error of the optimal portfolio diverges. In reference [5] we provided an analytic study of this phase transition, under the expected shortfall risk measure, based on the replica method [3] of statistical physics. In this note we use the same method but we extend it in two respects: we show how to extend it to other risk measures, and we study more realistic distributions of returns in which there is volatility clustering.

The paper is organized as follows. In Section 2 we introduce the notations we will use throughout the paper and we formulate the problem in its general mathematical form. In Section 3 we consider the case of the absolute deviation (AD) [6]. The phase transition induced from the noise estimation of the risk measure was studied in this case in [7] for the first time. We present the replica calculation of the optimal portfolio and compute explicitely a noise sensitivity measure introduced in reference [11]. In Section 4 we deal with portfolio optimization under Expected Shortfall [2,8], which was shown to have a non-trivial phase diagram [4] and then studied analytically [5]. The striking point is that, for some values of the external parameters of the problem, the minimization problem is not well defined and thus cannot admit a finite solution. We investigate here the same feature while

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considering realistic distribution of returns, so as to take into account volatility clustering. The replica approach then turns into a semi-analytic and extremely versatile technique. We discuss this point and then summarize our results in Section 5.

2 The general setting

We denote our portfolio by $\tilde{\mathbf{w}} = \{\tilde{w}_1, \ldots \tilde{w}_N\}$, where $\tilde{w}_i$ is the position on asset i. We do not impose any restriction to short selling: $\tilde{w}_i$ is a real number. The global constraint induced by the total budget reads $\sum_i \tilde{w}_i = W$. It is convenient to work in units of the available budget per asset, by using the variables $w_i = \tilde{w}_i N/W$. The budget constraint is then $\sum_i w_i = N$. We denote by x_i the return of the asset i and assume the existence of a well-defined probability density function (pdf) $\mathsf{p}(x_1, \ldots x_N)$. In practice, the mean return of the assets is much smaller than the volatility; for simplicity we just neglect the return here (but our method could be extended to impose some constraints on the expected return). The loss associated with a given portfolio is $\tilde{\ell} = \ell W/N$, where $\ell = -\sum_{i=1}^N w_i x_i$. One is thus interested in computing some properties of the pdf $p_{\mathbf{w}}(\ell)$ of the rescaled loss ℓ.

In practice, one chooses a risk measure $\mathcal{F}_\lambda(\ell)$ (which may depend on some auxiliary parameter λ), and the risk is defined as the expectation value $\int d\ell p_{\mathbf{w}}(\ell)\mathcal{F}_\lambda(\ell)$. The actual $\mathsf{p}(x_1, \ldots x_N)$ is not known, and the expectation value must be estimated by time series coming from market observations or synthetically produced by numerical simulations. Let us assume that we know the time series of the return on a time interval T: $\{x_i^{(1)}, x_i^{(2)}, \ldots x_i^{(T)}\}$. The risk associated with a given portfolio $w_1, \ldots w_N$, with risk measure $\mathcal{F}_\lambda$, is:

$$\mathrm{risk}(\mathbf{w}; N, T, \lambda) = \frac{1}{T}\sum_{\tau=1}^{T} \mathcal{F}_\lambda\left[-\sum_{i=1}^{N} w_i x_i^{(\tau)}\right]. \quad (1)$$

The best known example of risk measure is of course the variance, as first suggested by Markowitz. In this case the risk function is obtained by taking $\mathcal{F}_\lambda(z) = z^2$ in (1). The evaluation of the variance implies an empirical evaluation of the covariance matrix σ_{ij} of the underlying stochastic process, and the extremely noisy character of any estimation of σ_{ij} has been underlined a few years ago [9,10]. However, recent studies [11,12] have shown that the effect of the noise on the actual portfolio risk is not as dramatic as one might have expected. More in detail, a direct measure of this effect was introduced and explicitely computed in the simplest case of $\sigma_{ij} = \delta_{ij}$. In the next section, we compute the same quantity as far as the absolute deviation of the loss is concerned.

In the statistical physics approach, one studies the limit $N, T \to \infty$, while $N/T \equiv 1/t$ is finite. One introduces the partition function at inverse temperature γ:

$$Z_\gamma^{(N)}[t, \lambda; \{x_i^{(\tau)}\}] = \int \prod_{i=1}^{N} dw_i \; e^{-\gamma N t \,\mathrm{risk}[\mathbf{w}; N, Nt, \lambda]} \; \delta\left(\sum_{i=1}^{N} w_i - N\right), \quad (2)$$

from which any observable will be computed. For instance, the optimal cost (i.e. the minimum of the risk function in (1)) is computed from

$$\begin{aligned} e(t, \lambda) &= \lim_{N\to\infty} \min_{\mathbf{w}} \mathrm{risk}[\mathbf{w}; N, Nt, \lambda] \\ &= \lim_{N\to\infty} \frac{1}{Nt} \lim_{\gamma\to\infty} \frac{-1}{\gamma} \log Z_\gamma^{(N)}[t, \lambda; \{x_i^{(\tau)}\}]. \end{aligned} \quad (3)$$

It turns out that this expression depends on the actual sample (the time series $\{x_i^{(\tau)}\}$) used to estimate the risk measure. When the time series is generated from a probability measure $\mathsf{p}(x_1, \ldots x_N)$, it is reasonable to assume that their exists a large deviation principle, so that the distribution of $e(t, \lambda)$ (with respect to the various time series instances) is narrowly distributed around its mean when $N \to \infty$. Therefore we need to compute this mean, which requires to average the logarithm of the partition function according to the pdf $\mathsf{p}(\{x_i^{(\tau)}\})$. The so-called replica method allows to simplifiy this task as follows. We compute $\mathbb{E}\left[Z^n\right]$ for integer n and *assume* we can analytically continue this result to real n: then $\mathbb{E}\left[\log Z\right] = \lim_{n\to 0}(\mathbb{E}\left[Z^n\right] - 1)/n$. This is the strategy that we are going to use in the next sections and that will allow to compute the optimal portfolio.

3 Replica analysis: absolute deviation

The absolute deviation measure $\mathrm{AD}\big[\mathbf{w}; N, T\big]$ is obtained by choosing $\mathcal{F}_\lambda(z) = |z|$ in (1). No other external parameters λ are present here. We assume a factorized distribution

$$\mathsf{p}\big[\{x_i^{(\tau)}\}\big] \sim \prod_{i,\tau} \exp\left(-\frac{N (x_i^{(\tau)})^2}{2\sigma_\tau^2}\right), \quad (4)$$

where the volatilities $\{\sigma_\tau\}$ are distributed according to a pdf which we do not specify for the moment. The form in (4) is not too unrealistic (though fat tails are definitely neglected) in that it corresponds to a multinormal distribution seen in the bases of the eigenstates. Following the replica method, we introduce n identical replicas of our portfolio and compute the average of Z^n:

$$\mathbb{E}\left[Z_\gamma^n(t)\right] \sim \int \prod_{a,b=1}^{n} dQ^{ab} d\hat{Q}^{ab} \times e^{N\sum_{a,b=1}^{n}(Q^{ab}-1)\hat{Q}^{ab} - \frac{N}{2}\mathrm{Tr}\log\hat{Q} - \frac{T}{2}\mathrm{Tr}\log Q + \sum_\tau \log A_\gamma(\{Q^{ab}\};\sigma_\tau)},$$

$$A_\gamma(\{Q^{ab}\};\sigma_\tau) = \int \prod_{a=1}^{n} du_\tau^a \exp\left\{ -\frac{1}{2\sigma_\tau^2}\sum_{ab}(Q^{-1})^{ab}u_\tau^a u_\tau^b - \gamma\sum_a |u_\tau^a|\right\}, \quad (5)$$

where we have introduced the overlap matrix

$$Q^{ab} = \frac{1}{N}\sum_{i=1}^{N} w_i^a w_i^b\ , \quad a,b = 1,\dots n, \quad (6)$$

as well as its conjugate $\hat{Q}^{ab}$, the Lagrange multipliers introduced to enforce (6). In the limit $N,T\to\infty$, $N/T = 1/t$ finite, the integral in (5) can be solved by a saddle point method. Due to the symmetry of the integrand by permutation of replica indices, there exists a replica-symmetric saddle point [3]: $Q^{aa} = q_1$, $Q^{ab} = q_0$ for $a\neq b$, and the same for $\hat{Q}^{ab}$. We expect the saddle point to be correct in view of the fact that the problem is linear. Under this hypothesis, which will be only justified a posteriori by a direct comparison to numerical data, the replicated partition function in (5) gets simplified into

$$\mathbb{E}\left[Z_\gamma^n(t)\right] \sim \int dq_0 \int d\Delta q \exp\left[Nn\ S_\gamma(q_0,\Delta q)\,(1 + \mathcal{O}(n))\right], \quad (7)$$

$$S_\gamma(q_0,\Delta q) = \frac{(1-t)q_0 - 1}{2\Delta q} + \frac{1-t}{2}\log\Delta q + t\ \frac{1}{T}\sum_\tau \frac{1}{n}\log A_\gamma(q_0,\Delta q;\sigma_\tau),$$

$$A_\gamma(q_0,\Delta q;\sigma_\tau) = \int \frac{ds}{\sqrt{2\pi q_0}} e^{-s^2/2q_0}\left[1 + n\int du\ e^{-\frac{u^2}{2\Delta q\sigma_\tau^2} + \frac{s\ u}{\Delta q\sigma_\tau} - \gamma|u|} + \mathcal{O}(n^2)\right],$$

where $\Delta q = q_1 - q_0$ and n is the number of replicas (which will eventually go to zero). We now assume that in the low temperature limit the overlap fluctuations are of order $1/\gamma$ and introduce $\Delta = \gamma\Delta q$. One can show that if Δ stays finite at low temperatures

$$\lim_{n\to 0}\lim_{\gamma\to\infty}\frac{1}{n\gamma}\log A_\gamma(q_0,\Delta/\gamma;\sigma_\tau) = \Delta^2\sigma_\tau^3\int_1^\infty ds\ e^{-s^2\sigma_\tau^2\Delta^2/2q_0}(1-s)^2. \quad (8)$$

For the sake of simplicity, we focus on the simple case $\sigma_\tau = 1\ \forall\tau$. In the $\gamma\to\infty$ limit, the saddle point equations for (7) are

$$\frac{1}{t} = \mathrm{erf}\left(1/\sqrt{2q_0'}\right), \quad (9)$$

$$\Delta = \left(2t\left[\frac{1-1/t}{2}q_0' + \sqrt{\frac{q_0'}{2\pi}}e^{-1/2q_0'} - \frac{(1+q_0')}{2}\left(1 - \mathrm{erf}\left(1/\sqrt{2q_0'}\right)\right)\right]\right)^{-1/2}, \quad (10)$$

where $q_0 = q_0'\Delta^2$. The minimum cost function, i.e. the average of equation (3), is found to be $e(t) = 1/\Delta$. Notice that (3) only admits a solution for $t\geq 1$. There is no solution to the minimization problem if the ratio of assets to data points, N/T, is smaller than 1. On the other hand, once this condition is fulfilled, the equation (10) gives a finite Δ at any $t>1$. The asymptotic behaviour of $e(t)$ can be worked out analytically: we introduce $\delta\equiv 1-1/t$ and consider the limit $\delta\ll 1$. This leads to

$$e(t) \simeq \sqrt{\frac{\delta}{-2\log\delta}}\left(1 - \frac{\log\left(-\frac{4}{\pi}\log\delta\right)}{4\log\delta}\right). \quad (11)$$

The full solution and a comparison with numerics are shown in Figure 1 (left). Numerical simulations take advantage of the linear programming reformulation of the problem (see [4] for instance). We extract a given instance of the minimization problem, i.e. a given realization of the time series $\{x_i^{(\tau)}\}$, according to (4). We then look for the optimal portfolio $\{w_i^*\}$ by evaluating the objective function on the vertices of the simplex obtained by the intersection of the linear constraints [13]. We finally average our results (e.g. the variance of the optimal portfolio, or the minimum risk) over $\mathcal{O}(10^2)$ realizations of the disorder.

We now address the issue of noise sensitivity, for which a measure was introduced in [11]. The idea is the following: assume one knows the true pdf of the loss $p_{\mathbf{w}}(\ell)$ (in a model based approach). Then one can compute the optimal portfolio $\mathbf{w}^{(0)}$ by minimizing the absolute deviation of ℓ. This gives a benchmark to study the performance of the portfolio $\mathbf{w}^*$ obtained by minimizing the risk (1) based on an empirical time series of returns. In order to compare their performance, one introduces the ratio q_K defined as

$$q_K^2(N,T) = \frac{\mathrm{AD}\left[\mathbf{w}^*;N,T\right]}{\mathrm{AD}\left[\mathbf{w}^{(0)};N\right]}. \quad (12)$$

This is the quantity which we have computed by the replica approach. In our calculation we have assumed a factorized Gaussian distribution of returns (extensions to more realistic cases will be presented in the next section) and it is straightforward to prove that in this case $q_K = \sqrt{\sum_{i=1}^N (w_i^*)^2}$. This corresponds in our language to $\sqrt{q_0} = \sqrt{q_0'}\Delta$, which diverges like $(1-1/t)^{-1/2}$ as $1/t\to 1^-$. Corrections to this leading behavior (which is instead the full shape of q_K in the variance minimization problem) are needed in order to reproduce the data (right panel of Fig. 1). The comparison with the Markowitz optimal portfolio (variance minimization) indicates that the AD measure is actually less stable to perturbations: a geometric interpretation of this result can be found in reference [4]. Beside this fact, the interesting result is then the existence of a well defined threshold value $t=1$ at which the estimation error becomes infinite. This is due to the divergence of the variance of the optimal portfolio in the regime $t<1$, where any minimization attempt is thus totally meaningless.

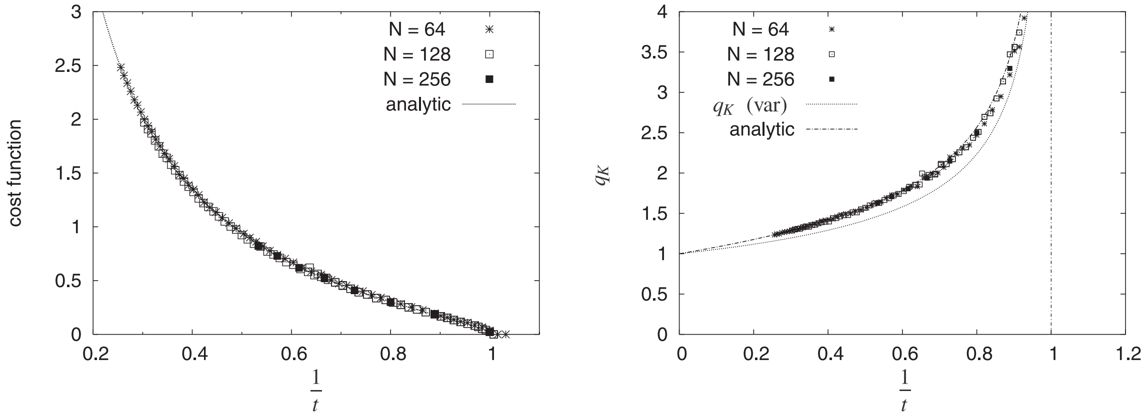

Fig. 1. Left: the analytic solution $e(t)$ is compared with the results of numerical simulations, where the constrained optimization is computed directly via linear programming methods [13]. Right: numerical results for $\sqrt{\sum_{i=1}^{N}(w_i^*)^2}$ compared to the analytic behaviour $\sqrt{q_0'}\Delta$. The curve denoted by q_K (var) represents the behaviour of q_K in the variance minimization problem.

4 Expected shortfall

4.1 The minimization problem

For a fixed value of $\beta < 1$ ($\beta \gtrsim 0.9$ in the interesting cases) the empirical estimation of the expected-shortfall risk measure is obtained by the minimization of a properly chosen objective function [14]:

$$\mathrm{ES}\big[\mathbf{w}; N, T, \beta\big] = \min_v \left\{ v + \frac{1}{(1-\beta)T} \sum_{\tau=1}^{T} \left[-v - \sum_{i=1}^{N} w_i x_i^{(\tau)} \right]^+ \right\}, \quad (13)$$

where $[a]^+ \equiv (a+|a|)/2$, and v is nothing but an auxiliary real variable (the value v^* that minimizes (13) could be interpreted as an approximation of the VaR: see [14] for more details). Optimizing the ES risk measure over all the possible portfolios satisfying the budget constraint is equivalent to the following linear programming problem:

- cost function: $E = (1-\beta)Tv + \sum_{\tau=1}^{T} u_\tau$;
- variables: $\mathbf{Y} \equiv \{w_1, \dots w_N, u_1, \dots u_T, v\}$;
- constraints: $u_t \geq 0, \quad u_t + v + \sum_{i=1}^{N} x_{it} w_i \geq 0, \quad \sum_{i=1}^{N} w_i = N$.

The intermediate step to understand the way this problem has been recast consists in replacing the $[\cdot]^+$ function by a variable u which can in principle be larger (this is guaranteed by the two inequalities in the constraints) but over which one has to minimize.

In a previous work [5] we solved the problem in the case where the historical series of returns is drawn from the oversimplified probability distribution (4), with $\sigma_\tau = 1 \ \forall \tau$. Here we do a first step towards dealing with more realistic data and assume that the series of returns can be obtained by a sequence of normal distributions whose variances depend on time:

$$p\big[\{\sigma_t\}\big] \sim \prod_{\tau,\tau'} \exp\left(-\sigma_\tau \sigma_{\tau'} G^{-1}_{\tau,\tau'}\right) \prod_{\tau} q(\sigma_\tau), \quad (14)$$

for some long range correlator $G_{\tau,\tau'}$ which takes into account volatility correlations, and $q(\sigma_\tau)$ equal e.g. to a log-normal distribution.

4.2 The replica solution

A straightforward generalization of the replica calculation presented in reference [5] (and sketched in the previous section for a similar problem) allows to compute the average optimal cost for a given volatility sequence $\{\sigma_1, \dots \sigma_T\}$, in the limit when $N, T \to \infty$ and $N/T = 1/t$ stays finite. This is given by

$$e(t,\beta) = \min_{v,q_0,\Delta} \left[\frac{1}{2\Delta} + \Delta \, \tilde{\varepsilon}(t,\beta; v, q_0 | \{\sigma_\tau\}) \right], \quad (15)$$

$$\tilde{\varepsilon}(t,\beta; v, q_0|\{\sigma_\tau\}) \equiv t(1-\beta)v - \frac{q_0}{2} + \frac{t}{2\sqrt{\pi}} \frac{1}{T} \sum_{\tau=1}^{T} \int_{-\infty}^{+\infty} ds \ e^{-s^2} g(v/\sigma_\tau + s\sqrt{2q_0}; \sigma_\tau), \quad (16)$$

where $\Delta \equiv \lim_{\gamma\to\infty} \gamma \Delta q$ and the function $g(x;\sigma)$ is equal to x^2 if $-\sigma \leq x < 0$, to $-2\sigma x - \sigma^2$ is $x < -\sigma$, and 0 otherwise. The minimization over v, q_0 implies that

$$\partial\tilde{\varepsilon}/\partial v = \partial\tilde{\varepsilon}/\partial q_0 = 0. \quad (17)$$

As discussed in [5], the problem admits a finite solution if (16) is minimized by a finite value of Δ. The feasible region is then defined by the condition $\tilde{\varepsilon}(t,\beta; v, q|\{\sigma_t\}) \geq 0$, where v and q_0 satisfy (17). This theoretical setup suggests the following semi-analytic protocol for determining the phase diagram of realistic portfolio optimization problems.

1. Fix a value of $\beta \in [0,1]$, and take N equal to the portfolio size you are interested in.

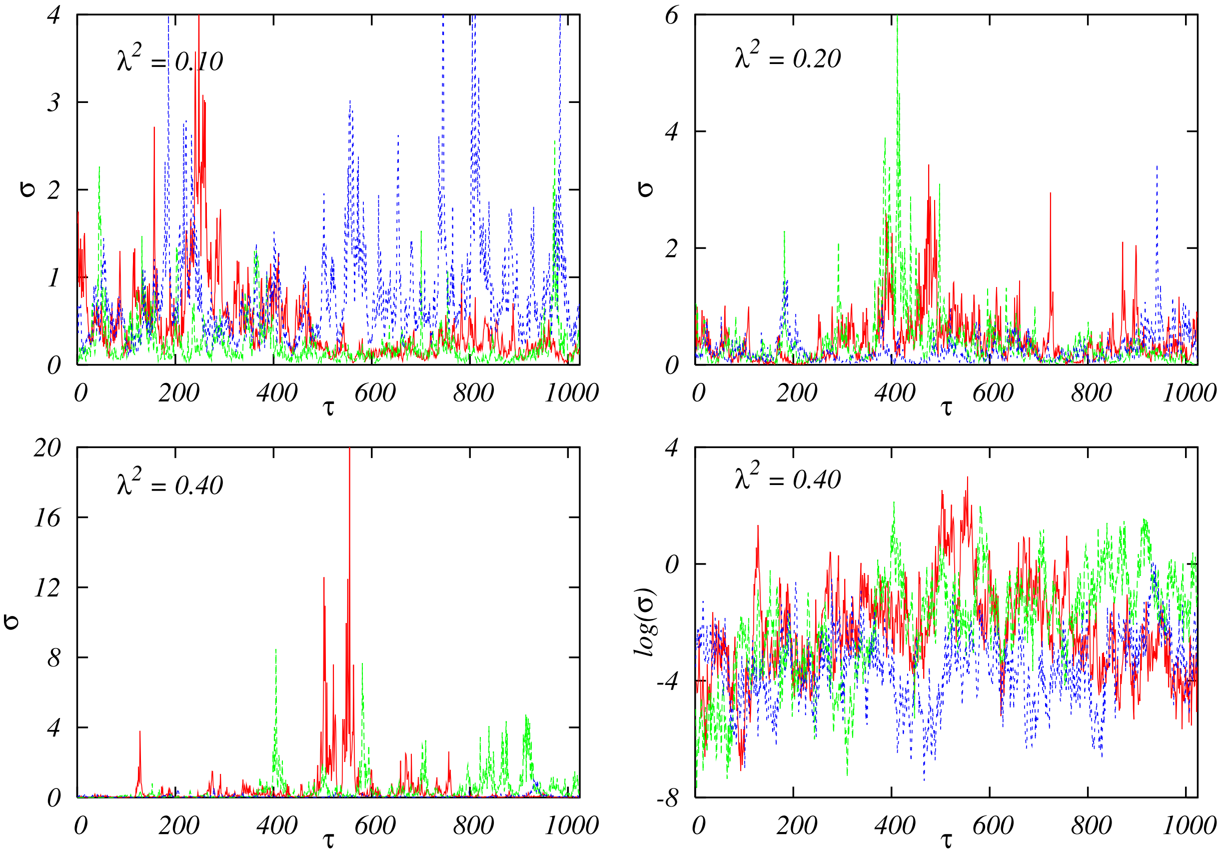

Fig. 2. The first three panels show 3 realizations of volatility sequences of length $T = 1024$ according to the model (18). Different panels correspond to different values of λ^2. The last panel is a logarithmic representation of the $\lambda^2 = 0.40$ data.

2. For $T = T_{\min}$ to $T_{\max}$, such that $N/T \in [0.1, 0.9]$, do the following:
   (a) generate a sequence $\{\sigma_1, \sigma_2, \ldots \sigma_T\}$ according to (14) and compute the $\tilde{\varepsilon}$ function in (16);
   (b) minimize $\tilde{\varepsilon}$ with respect to v and q_0 according to (17);
   (c) repeat steps (a) and (b) for n samples, and compute the mean value $\langle \tilde{\varepsilon} \rangle$.
3. Plot $\langle \tilde{\varepsilon} \rangle$ vs. N/T and find the value $(N/T)^*$ where this function changes its sign.

By repeating this procedure for several values of β we get the phase separation line $(N/T)^*$ vs. β.

4.3 Results

A simple way of generating realistic volatility series consists in looking at the return time series as a cascade process [15]. In a multifractal model recently introduced [16] the volatility covariance decreases logarithmically: this is achieved by letting $\sigma_\tau = \exp \xi_\tau$, where ξ_τ are Gaussian variables and

$$\langle \xi_\tau \rangle = -\lambda^2 \log T_{\text{cut}} , \quad \langle \xi_\tau \xi_{\tau'} \rangle - \langle \xi_\tau^2 \rangle = \lambda^2 \log \frac{T_{\text{cut}}}{1 + |\tau - \tau'|}, \tag{18}$$

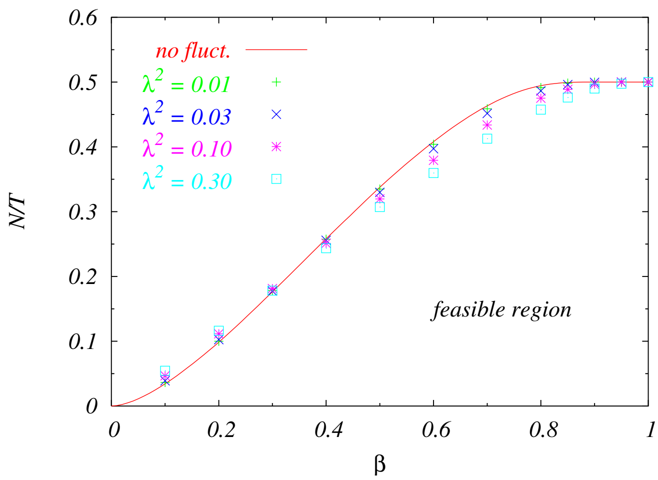

Fig. 3. The phase diagram corresponding to different values of the parameter λ^2. The full line corresponds to the absence of fluctuations in the volatility distributions (i.e. $\sigma_\tau = 1\ \forall \tau$).

λ quantifying volatility fluctuations (the so-called 'vol of the vol'), and T_{cut} being a large cutoff. A few samples generated according to this procedure are shown in Figure 2.

The phase diagram obtained for different values of λ^2 is shown in Figure 3. A comparison with the phase diagram computed in absence of volatility fluctuations shows that, while the precise shape of the separating curve depend on the fine details of the volatility pdf, the main message has not changed: there exists a regime, $N/T > (N/T)^*$, where the small number of data with respect to the portfolio size makes the optimization problem ill-defined. In the "max-loss" limit $\beta \to 1$, where the single worst loss contributes to the risk measure, the threshold value $(N/T)^* = 0.5$ does not seem to depend on the volatility fluctuations. As β gets smaller than 1, though, the presence of these fluctuations is such that the feasible region becomes smaller than the ideal multinormal case.

5 Conclusions

In this paper we have discussed the replica approach to portfolio optimization. The rather general formulation of the problem allows to deal with several risk measures. We have shown here the examples of absolute deviation, expected shortfall and max-loss (which is simply taken as the limit case of ES). In all cases we find that the optimization problem, when the risk measure is estimated by using time series, does not admit a feasible solution if the ratio of assets to data points is larger than a threshold value. As discussed in reference [4], this is a common feature of various risk measures: the estimation error on the optimal portfolio, originating from in-sample evaluations, diverges as a critical value is approached. In the expected shortfall case, we have also discussed a semi-analytic approach which is suitable for describing realistic time series. Our results suggest that, as far as volatility clustering is taken into account, the phase transition is still there, the only effect being the reduction of the feasible region. As a general remark, we have shown that the replica method may prove extremely useful in dealing with optimization problems in risk management.

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